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A remarkable class of non-classical symmetries of the steady two-dimensional boundary-layer equations

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Abstract
We study the steady two-dimensional boundary-layer equations in the flat and axisymmetric case to show that some similarity reductions already found in the literature with *ad hoc* new methods of reduction are indeed invariant solutions under the action of non-classical symmetries (as introduced by Bluman and Cole). Moreover, we show that the celebrated von Mises transformation that reduces the two-dimensional flat boundary-layer equations to a second-order evolution equation is indeed a Bäcklund transformation related to a non-classical symmetry.

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1. Introduction

The classical method of finding similarity reductions of partial differential equations based on the computation of classical Lie symmetries is well known (see, for example, Olver 1993). To apply this method to a partial differential equation in two independent variables

$$\Delta(x, y, u, u^{(k)}) = 0 \tag{1}$$

with $u^{(k)}$ denoting the derivatives of the unknown function u with respect to the x and y up to the order k, we consider the one-parameter (ϵ) Lie group of infinitesimal transformations in (x, y, u) given by

$$\begin{aligned} x^* &= x + \epsilon\xi(x, y, u) + O(\epsilon^2) \\ y^* &= y + \epsilon\tau(x, y, u) + O(\epsilon^2) \\ u^* &= u + \epsilon\eta(x, y, u) + O(\epsilon^2) \end{aligned} \tag{2}$$

which leaves (1) invariant. The generators, ξ, η and τ, of (2) are determined from the infinitesimal invariance requirement

$$(\mathbf{v}^{(k)}\Delta)|_{\Delta=0} = 0 \tag{3}$$

where $\mathbf{v}^{(k)}$ is the usual kth prolongation of the transformation group (Olver 1993).

Having defined a symmetry group, the corresponding similarity reduction and therefore the invariant solutions may be obtained from the overdetermined system, $\mathcal{S}$, composed of equation (1) and the first-order quasi-linear equation denoted as the *invariant surface condition*

$$Q(x, y, u, u^{(1)}) := \eta(x, y, u) - \xi(x, y, u)u_x - \tau(x, y, u)u_y = 0 \tag{4}$$

i.e.

$$\mathcal{S} := \begin{cases} \Delta = 0 \\ Q(x, y, u, u^{(1)}) = 0. \end{cases} \tag{5}$$

It has been shown by Pucci and Saccomandi (1992) (see also, Seiler 1997) that the reduction method based on the classical Lie groups is only a special step in the study of the full *compatibility problem* for the overdetermined system $\mathcal{S}$, and that the complete study of this compatibility problem allows us to recover not only the similarity reductions corresponding to classical symmetries, but also the similarity reductions corresponding to the *non-classical symmetries* introduced by Bluman and Cole (1969) and to the *weak symmetries* introduced by Olver and Rosenau (1986). It is important to point out that only when ξ, η and τ are associated with classical or non-classical symmetries of (1) the solutions of $\mathcal{S}$ may be obtained by reduction of $\Delta = 0$ to a *single* ordinary differential equation; when ξ, η and τ are associated with weak symmetries, we reduce the given equation to an overdetermined, but compatible, system of ordinary differential equations. The use of the theory of first-order quasi-linear differential equations and the findings about the compatibility of $\mathcal{S}$ by Pucci and Saccomandi (1992) has been very useful to show when the *direct methods* of reductions (as the one by Clarkson and Kruskal (1988) and the other by Rubel (1991)) are equivalent to classical, non-classical or weak symmetries (see Pucci 1992, Pucci and Saccomandi 1995, Saccomandi 1997, Pucci and Saccomandi 2000).

Surveys about non-classical and weak symmetries may be found in Olver and Vorobev (1994) and Clarkson (1995). Moreover, in the introduction of Pucci and Saccomandi (2000), it is possible to find a detailed discussion on the relationship between the compatibility of the overdetermined system (5), the symmetries of (1) and direct methods.

The aim of this paper is to show, by examples, that the complete and careful study of the compatibility problem of the overdetermined system $\mathcal{S}$ seems to be the more efficient and simpler way to study the problem of the determination of similarity solutions. The example that we consider is given by the equations of the stationary two-dimensional boundary-layer (BL) approximation to the full Navier–Stokes equations.

In the *flat* and *axisymmetric* case, the classical symmetries of BL equations have been considered long time ago by Ovsiannikov (1982), and it is well known that this invariance group is quite rich. For the case of the flat BL equations, non-classical symmetries have been computed in a very interesting paper by Burde (1996), but as declared by the author these computations are *not* complete. Indeed, Burde does not consider the special case where one of the infinitesimal generators associated with an independent variable is identically zero. Here we show that this special case is related to an interesting and, to the best of our knowledge, unnoticed property: the celebrated von Mises transformation (von Mises 1927) may be deduced by the symmetry analysis. On the other hand, for the axisymmetric BL equations in Burde (1994) some exact solutions are obtained by a direct reduction method. This method consists in a very interesting modification of the method introduced by Clarkson and Kruskal (1988), a modification which shares some similarities with the method of quasi-solutions by Rubel (1991). In the last section of Burde (1994), the author *shows* that the solution obtained via the direct reduction method cannot be obtained as similarity solutions associated with a classical (see p 256 of Burde (1994)) or non-classical (see p 257 of Burde (1994)) Lie group. In Goard and Broadbridge (1999) the new method *of symmetry-enhancing constraints* is suggested, and it is shown that by this method it is possible to find all the solutions considered in Burde (1994) as invariant solutions. Goard and Broadbridge define the symmetry-enhancing constraints 'as equations which when added to a target equation, result in the *enlarged system* having at least one additional symmetry not possessed by the target equation on its own' (Goard and Broadbridge 1999, p 369). The enlarged system is in reality an overdetermined system and therefore being a solution space of the enlarged system of a subset of the solution space of the target equation; there is a hope to find new symmetries. Although, generally speaking, the overdetermined enlarged system is more general than the system (5), we show that in the case of the examples reported in Goard and Broadbridge (1999) for the axisymmetric BL equations we always consider special classes of non-classical symmetries. All these non-classical symmetries exactly of the kind are not considered in Burde (1996) and skipped in Burde (1994, p 256 and 257).

The plan of the paper is the following: in the next section we introduce the basic equations, in section 3 we study the flat BL equations, whereas the axisymmetric case is studied in section 4 and concluding remarks are devoted to the last section.

2. Basic equations

The boundary-layer equations are a standard approximation of the full Navier–Stokes equations introduced to study the viscous flow over a surface at very high Reynolds numbers. If we consider the special case where the surface past which the liquid flows is a flat plate in the two-dimensional steady case, the boundary-layer equations are given by the system

$$\begin{cases} u_x + v_y = 0 \\ uu_x + vu_y = u^{(e)}u_x^{(e)} + \nu u_{yy}. \end{cases} \tag{6}$$

Here subscripts are used to denote partial differentiation, x and y are the orthogonal Cartesian coordinates parallel and perpendicular to the plate ($y = 0$), u and v are the longitudinal and transverse components of the fluid velocity, ν is the fluid kinematic viscosity and $u^{(e)}(x, t)$ is the given external flow such that

$$\lim_{y\to\infty} u = u^{(e)}. \tag{7}$$

By rescaling by ν, the two components of the velocity, i.e. $u \to \nu u$, ($u^{(e)} \to \nu u^{(e)}$) and $v \to \nu v$, equation $(6)_1$ may be rewritten with the kinematic viscosity set equal to 1.

On the other hand, if we consider the case of an elongated slender body of revolution in a longitudinal flow it is convenient to introduce cylindrical coordinates (x, r). Here the x-axis coincides with the axis of the body. In this case, the boundary-layer equations take the form

$$\begin{cases} u_x + v_r + \frac{1}{r}v = 0 \\ uu_x + vu_r = u^{(e)}u_x^{(e)} + \nu\left(u_{rr} + \frac{1}{r}u_r\right). \end{cases} \tag{8}$$

By introducing the stream function $\psi(x, y, t)$, we rewrite the boundary-layer equations (6) as the single third-order partial differential equation

$$\psi_{yyy} + \psi_x\psi_{yy} - \psi_y\psi_{xy} - \Theta_x = 0 \tag{9}$$

where

$$\Theta(x) = \tfrac{1}{2}u^{(e)2}(x). \tag{10}$$

We point out that Alassia and Nucci (1996) have considered the system (6) in the case $\Theta \equiv 0$, and they have reduced this system to a single partial differential equation

$$u_y u_{yyy} - u u_y u_{xy} + u u_x u_{yy} - u_{yy}^2 = 0 \tag{11}$$

that they call the Prandtl equation. This equation is obtained by differentiation of $(6)_2$ with respect to y and straightforward manipulations. The choice of Alassia and Nucci (1996) is not connected to the physics of the problem, and (11) is more complicated than (9). In their paper, they find some solutions of (11) by using a heavy computational method based on iterations of the non-classical method. The physical significance of these solutions is not discussed by the authors.

It is also possible to extend equation (9) in the framework of non-Newtonian fluid mechanics. In this case

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} + \Theta_x = \frac{\partial \mathcal{F}(\psi_{yy})}{\partial y} \tag{12}$$

where $\mathcal{F}(\psi_{yy})$ is the non-Newtonian stress tensor contribution. For example, if we are considering a power-law fluid we have that

$$\mathcal{F}(\psi_{yy}) = \frac{k}{n}(\psi_{yy})^n \tag{13}$$

with k the non-Newtonian viscosity and n a power-law parameter (Polyanin 2001).

On the other hand, introducing the stream function $\psi(x, \mu, t)$, where $\mu = r^2/4$, it is possible to rewrite the boundary-layer equations (8) as

$$\psi_\mu \psi_{\mu x} - \psi_x \psi_{\mu\mu} - 4\Theta_x = 2(\mu \psi_{\mu\mu\mu} + \psi_{\mu\mu}). \tag{14}$$

Here ν has been set equal to 1 by the same rescaling that we have already considered in the flat case.

We advice the reader that in Burde (1996) the author has changed the notation to make it similar to the one used in group theoretical considerations; here we prefer not to perform this change of notation.

3. Flat boundary-layer equations

Our starting point is the overdetermined system

$$\mathcal{S}_F : \begin{cases} \psi_{yyy} + \psi_x \psi_{yy} - \psi_y \psi_{xy} - \Theta_x = 0 \\ \psi_y - \eta(x, y, \psi) = 0. \end{cases} \tag{15}$$

The $(15)_2$ is the invariant surface condition corresponding to the infinitesimal associated with the x variable sets to zero, and the infinitesimal associated with the variable y sets equal to 1 (and this without loss of generality). This is a missing case of Burde (1996).

To study the compatibility problem, first of all we consider the following differential consequences of $(15)_2$:

$$\begin{cases} \psi_{yy} = \eta_y + \eta\eta_\psi \\ \psi_{xy} = \eta_x + \eta_\psi \psi_x \\ \psi_{yyy} = \eta_{yy} + \eta_y \eta_\psi + \eta^2 \eta_{\psi\psi} + 2\eta\eta_{y\psi} + \eta\eta_\psi^2. \end{cases} \tag{16}$$

The introduction of (16) into (15) reduces $\mathcal{S}_F$ to the following first-order system:

$$\begin{cases} \eta_y \psi_x = \Theta_x - \eta_{yy} - \eta_\psi \eta_y - \eta^2 \eta_{\psi\psi} - 2\eta\eta_{y\psi} - \eta\eta_\psi^2 + \eta\eta_x \\ \psi_y = \eta(x, y, \psi) \end{cases} \tag{17}$$

and now the original compatibility problem may be studied using the standard Lagrange–Charpit method (Courant and Hilbert 1937). In doing so two cases have to be discussed separately: $\eta_y \equiv 0$ and $\eta_y \neq 0$.

We point out that because we are considering a first-order system it is clear that the possibility to obtain a solution by reduction of the BL equations stops to the first step of the full compatibility problem. This means that weak symmetries cannot exist in the class of invariant surface conditions considered here. Indeed, the compatibility relation of the system (17), which we shall obtain by a straightforward cross-differentiation of the first-order derivatives, does not contain the derivatives of the unknown function ψ. This is an happenstance due to the special mathematical structure of the overdetermined system.

3.1. The von Mises transformation

First of all we discuss the *degenerate* case $\eta_y \equiv 0$. Now, $(17)_1$ reduces to

$$\Theta_x - \eta^2\eta_{\psi\psi} - \eta\eta_\psi^2 + \eta\eta_x = 0. \tag{18}$$

The (18) is an *evolution* equation in the unknown $\eta(x, \psi)$. Given a solution $\widetilde{\eta}$ of this *evolution* equation by solving the first-order ordinary differential equation $(17)_2$, we find

$$\int \frac{\mathrm{d}\psi}{\widetilde{\eta}(x, \psi)} = y + g(x). \tag{19}$$

The (17) is an implicit solution of the BL equations. On the other hand, if we consider any solution $\psi = \psi(x, y)$ of the BL equations (locally) it is possible to define

$$y = y(x, \psi) \tag{20}$$

and to compute

$$\frac{\partial y}{\partial \psi} = \frac{1}{\psi_y} \qquad \frac{\partial y}{\partial x} = \frac{\psi_x}{\psi_y}. \tag{21}$$

Because $(17)_2$ is in force we have

$$\begin{cases} \eta_\psi = \frac{\psi_{yy}}{\psi_y} \\ \eta_{\psi\psi} = \left(\frac{\psi_{yy}}{\psi_y}\right)_\psi \frac{1}{\psi_y} \\ \eta_x = \psi_{xy} - \frac{\psi_x}{\psi_y}\psi_{yy} \end{cases} \tag{22}$$

and introducing (22) into (18) we recover

$$\Theta_x - \psi_{yyy} + \psi_y\psi_{yx} - \psi_x\psi_{yy} \equiv 0. \tag{23}$$

The (23) is *identically* satisfied because we have, by hypothesis, that $\psi(x, y)$ is a solution of the BL equation.

This is exactly the von Mises transformation (von Mises 1927). The idea that via non-classical symmetries it is possible to recover Bäcklund transformations; it is not new. To the best of our knowledge this fact has been recorded for the first time in Nucci (1993) by some classical and well-known examples. Our approach based on the compatibility of the overdetermined system $\mathcal{S}$ clarifies the nature of this connection. We point that in Alassia and Nucci (1996) there was no possibility of recovering this interesting Bäcklund transformation because they use an exotic reduction of the system (6) to a single equation.

The result considered here is interesting because it provides a simple and algorithmic method to search the von Mises transformation for any kind of differential equations. For example, the characterization of this transformation via non-nonclassical symmetries, in the framework of non-Newtonian fluids, gives immediately the following target evolution equation:

$$\eta\eta_x + \Theta_x = -\mathcal{F}'\big(\eta^2\eta_{\psi\psi} + \eta\eta_\psi^2\big) \tag{24}$$

where the prime denotes differentiation with respect to an argument of the function $\mathcal{F}$. This is an important result because we know only few similarity reductions for non-Newtonian BL equations.

We point out that (18) when $\Theta_x = 0$ reduces to

$$\eta_x = \eta\eta_{\psi\psi} + \eta_\psi^2 \tag{25}$$

and this is a special case of the remarkable diffusion equation studied in detail by King (1993). This happenstance allows us to have a long list of exact solutions already computed to be transformed by the Bäcklund transformation considered here to exact solutions of the BL equations. For example, if we consider the instantaneous source solution of equation (25), i.e.

$$\eta(x, \psi) = kx^{-1/3} - \tfrac{1}{6}\psi^2 x^{-1} \tag{26}$$

where k is an arbitrary constant; by using the Bäcklund transformation we obtain, when $k = h^2$, the following exact solutions for the BL equations:

$$\psi(x, y) = \frac{h\sqrt{6}x^{1/3}\left(\exp\left(\frac{h\sqrt{6}(y+g(x))}{3x^{2/3}}\right) + 1\right)}{\exp\left(\frac{h\sqrt{6}(y+g(x))}{3x^{2/3}}\right) - 1}. \tag{27}$$

Here $g(x)$ is an arbitrary function appearing in (19). It may be easily checked that the condition (7) is satisfied.

Obviously, the Bäcklund transformation works also when the far-field exterior flow velocity is *not* zero. A simple example is given by considering the trivial solution of the evolution equation (18) given by

$$\eta^2(x, \psi) = k\psi + g(x) \tag{28}$$

where k is an arbitrary constant and $g(x)$ is a function determined by the far-field exterior flow θ_x via the relation $2\theta_x + g_x = 0$. Using (28) we find the exact solution of the BL equations

$$\psi(x, y) = \frac{k^2(y + f(x))^2 - 4g(x)}{4k}$$

where $f(x)$ is an arbitrary function. This solution is valid for any far-field exterior flow, but does not fulfil the condition (7).

3.2. The general case

When $\eta_y \neq 0$, it is possible to obtain the integrability condition of $\mathcal{S}_F$ by cross-differentiation of $(17)_1$ and $(16)_2$

$$0 = D_y\left\{\frac{\Theta_x - \eta_{yy} - \eta_\psi\eta_y - \eta^2\eta_{\psi\psi} - 2\eta\eta_{y\psi} - \eta\eta_\psi^2 + \eta\eta_x}{\eta_y}\right\} - \eta_\psi\left\{\frac{\Theta_x - \eta_{yy} - \eta_\psi\eta_y - \eta^2\eta_{\psi\psi} - 2\eta\eta_{y\psi} - \eta\eta_\psi^2 + \eta\eta_x}{\eta_y}\right\} - \eta_x \tag{29}$$

where after performing the total derivative we have to replace $\psi_y = \eta(x, y, \psi)$. Here D_y is the usual total derivative operator with respect to the variable y.

The solution of the integrability relation (29) is more complicated than the solution of the BL equations. This is a standard problem when we consider non-classical and weak symmetries of partial differential equations: the determining equations are nonlinear equations also in the case of a linear partial differential equation (1). In any case, we point out that a trivial solution of (29) may be related to a nontrivial solution of the BL equations, and if we consider special forms of $\eta(x, y, \psi)$ also a set of complex determining equations as (29) may be easily solved. This is, for example, the case when the integrability conditions may be rewritten in a *separated form*. We recall that an equation is in a separated form when it is written as

$$\sum_{i=1}^{m} \Lambda_i(\chi)\Gamma_i(\mu) = 0 \tag{30}$$

where the $\Lambda_i(\chi)$ are expressions regarding the variables $\chi = \chi_1, \ldots, \chi_s$ and functions of these variables, whereas the $\Gamma_i(\mu)$ are expressions regarding a set of different variables $\mu = \mu_1, \ldots, \mu_r$, and functions of these variables. The occurrence of equations (30) in group analysis has been discussed by Pucci and Saccomandi (2000), and we refer to this paper for the details. We remark that, at the best of our knowledge, in all the nontrivial applications where the non-classical symmetries have been computed explicitly the determining equations were always in the form (30).

For the above-mentioned reasons, we consider the solution of (29) when

$$\eta(x, y, \psi) = f(x, y)\Phi(\psi). \tag{31}$$

The introduction of this ansatz in (29) gives the following equation in a separated form:

$$\begin{aligned}\Phi\left(\Phi^2\frac{\mathrm{d}^3\Phi}{\mathrm{d}\psi^3} + 2\Phi\frac{\mathrm{d}\Phi}{\mathrm{d}\psi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\psi^2} - \left(\frac{\mathrm{d}\Phi}{\mathrm{d}\psi}\right)^3\right)\mathcal{A}_7 + \Phi^2\frac{\mathrm{d}^2\Phi}{\mathrm{d}\psi^2}\mathcal{A}_6 + \Phi\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\psi}\right)^2\mathcal{A}_5 \\ + \Phi\frac{\mathrm{d}\Phi}{\mathrm{d}\psi}\mathcal{A}_4 + \frac{\mathrm{d}\Phi}{\mathrm{d}\psi}\mathcal{A}_3 + \Phi^2\mathcal{A}_2 + \Phi\mathcal{A}_1 + \mathcal{A}_0 = 0\end{aligned} \tag{32}$$

where

$$\begin{aligned}
\mathcal{A}_0 &= \frac{\partial^2 f}{\partial y^2}\Theta_x \\
\mathcal{A}_1 &= \frac{\partial f}{\partial y}\frac{\partial^3 f}{\partial y^3} - \left(\frac{\partial^2 f}{\partial y^2}\right)^2 \\
\mathcal{A}_2 &= f\left(\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x\partial y}\frac{\partial f}{\partial y}\right) \qquad \mathcal{A}_3 = f\frac{\partial f}{\partial y}\Theta_x \\
\mathcal{A}_4 &= \frac{\partial f}{\partial y}\left(3\left(\frac{\partial f}{\partial y}\right)^2 - f\frac{\partial^2 f}{\partial y^2}\right) \qquad \mathcal{A}_5 = -f^3\frac{\partial^2 f}{\partial y^2} \\
\mathcal{A}_6 &= f^2\left(6\left(\frac{\partial f}{\partial y}\right)^2 - f\frac{\partial^2 f}{\partial y^2}\right) \qquad \mathcal{A}_7 = f^4\frac{\partial f}{\mathrm{d}y}.
\end{aligned} \tag{33}$$

In solving equation (32) nontrivial solutions (i.e. solutions for which $\eta_y \neq 0$ and $\eta_\psi \neq 0$) are possible if and only if we have that $\Phi(\psi) = \psi$ or $\Theta(x) = U$.

When $\Phi(\psi) = \psi$, we find only one nontrivial case where

$$f(x, y) = \frac{1}{y + \beta(x)} \tag{34}$$

and therefore we recover the following class of solutions:

$$\psi(x, y) = \alpha(x)(y + \beta(x)) \tag{35}$$

where it must be

$$\Theta_x = -\alpha\alpha_x. \tag{36}$$

This family of exact solutions, which depend on an arbitrary function of the independent variable x, i.e.

$$\psi(x, y) = -u^{(e)}(x)(y + \beta(x)). \tag{37}$$

It is not interesting from the point of view of physics because the corresponding longitudinal component of the fluid velocity does not depend on y, and therefore a boundary layer structure is not possible.

On the other hand, when $\Theta(x) = U$, we recover the general solution

$$\eta(x, y, \psi) = \frac{2\Phi(\psi)}{k_1 y + \beta(x)} \tag{38}$$

where k_1 is an arbitrary constant and $\Phi(\psi)$ satisfies the ordinary differential equation

$$4(\Phi^2\Phi'')' = 8k_1\Phi\Phi'' + 4\Phi'^3 - 4k_1\Phi'^2 - k_1^2\Phi' - k_1^3. \tag{39}$$

For the nonlinear equation (39) is possible to obtain the following two parameters, (a, b), family of exact solutions:

$$\Phi(\psi) = a\psi^2 + b\psi - \left(\frac{k_1^2 - 4b^2}{16a}\right). \tag{40}$$

Introducing (40) into (38) from $(15)_2$, we obtain the similarity ansatz

$$\begin{aligned}\psi_\pm(x, y) = {} & \left\{k_1^2(k_1 + 2b)y^2 + 2k_1(k_1\beta(x) \pm k_1\sqrt{\alpha(x)} + 2b\beta(x))y\right.\\ & \left. + (\beta^2(x) \pm \sqrt{\alpha(x)})^2 + 2b(\beta^2(x) - \alpha(x))\right\}\\ & \times [4a(\alpha(x) - (k_1 y + \beta(x))^2)]^{-1}\end{aligned} \tag{41}$$

where $\alpha(x)$ is the similarity function to be determined. Indeed, introducing (41) in (23) we obtain the similarity reductions

$$\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x} \pm 24a\sqrt{\alpha(x)} = 0 \to \sqrt{\alpha(x)} = \mp\frac{21}{2}a(x + k_2) \tag{42}$$

where k_2 is an integration constant. The exact solution (41), to the best of my knowledge, seems to have not been recorded elsewhere, and it may be easily checked that the condition (7) is satisfied when $U = 0$.

Obviously the choice (31) is not the only possibility; for example, considering

$$\eta(x, y, \psi) = \Gamma(\psi) + f(y) + g(x) \tag{43}$$

the general solution of (29) is given by

$$\Gamma(\psi) = k\psi \qquad f(y) = k_1 y \qquad \Theta(x) = U \tag{44}$$

where k, k_1 and U are constants. The integration of (15) in correspondence with (43) and (44) gives

$$\psi(x, y) = \alpha(x)\exp(ky) - g(x)/k - k_1(ky + 1)/k^2 \tag{45}$$

where $\alpha(x)$ is the function to be determined by the reduction of the BL equations. By the introduction of (45) into (9), we obtain the ordinary differential equation

$$k_1\frac{\mathrm{d}\alpha}{\mathrm{d}x} + \left(k^3 - k\frac{\mathrm{d}g}{\mathrm{d}x}\right)\alpha = 0 \tag{46}$$

whose general solution is

$$\alpha(x) = \exp\left(-\frac{k^3 x + k_2 - g(x)k}{k_1}\right) \tag{47}$$

with k_2 is the integration constant. If we rename the various constant in (48) as

$$k = -\lambda \qquad k_1 = c\lambda \tag{48}$$

and the functions as

$$g(x) = c\lambda q(x) + R\beta(x) + c \qquad \alpha(x) = M\beta(x)\exp(-\lambda q(x)) \tag{49}$$

we generalize the similarity solution (3.34) of Burde (1996, p 1676). Indeed by inserting the (49) into (47), we obtain

$$\lambda^2 x = \frac{k_2}{\lambda} + \beta R + c\log(M) + c\log(\beta) + c \tag{50}$$

which is more general than the (3.34b) of Burde (1996). (Burde result is obtained, up to translation, considering $k_2 = 0$ and $M = 1$.)

We remark that Burde (1996) proposes the solution (3.34) as an example of a *new extension of the non-classical method* (see p 1671). Here we have shown that this solution is obtainable by the non-classical method if this algorithm is carefully implemented. The same result holds for the solution (3.35) in Burde (1996). It is only necessary to consider a more complex form of η, and to avoid long computations we skip the details of this computation.

Therefore the statement contained in Burde (1996, p 1677) (just before section 4) is wrong. All the examples of similarity solutions listed in this paper are obtainable by the standard non-classical method of Bluman and Cole.

4. Axisymmetric boundary-layer

Now the starting point is the overdetermined system

$$\mathcal{S}_A : \begin{cases} \psi_\mu \psi_{\mu x} - \psi_x \psi_{\mu\mu} - 2(\mu\psi_{\mu\mu})_\mu - 4\Theta_x = 0 \\ \psi_\mu - \eta(x, \mu, \psi) = 0 \end{cases} \tag{51}$$

which is equivalent, considering the differential consequences of $(51)_2$, to the first-order quasi-linear system

$$\begin{cases} \eta_\mu \psi_x = \eta\eta_x - 4\Theta_x - 2\eta_\mu - 2\eta\eta_\psi \\ \quad -2\mu\left(\eta^2\eta_{\psi\psi} + \eta\eta_\psi^2 + \eta_\mu\eta_\psi + \eta_{\mu\mu} + 2\eta\eta_{\mu\psi}\right) \\ \psi_\mu = \eta(x, \mu, \psi). \end{cases} \tag{52}$$

If $\eta_\mu = 0$, it must be

$$\eta^2(x, \psi) = k_1\psi + g(x) \tag{53}$$

where

$$\frac{\mathrm{d}g(x)}{\mathrm{d}x} = 8\Theta_x + 2k_1. \tag{54}$$

Integrating $(51)_2$ when (53) is taken into account gives

$$\psi(x, \mu) = \frac{k_1}{4}(\mu + g_1(x))^2 - \frac{g(x)}{k_1} \tag{55}$$

and it is possible to check directly that (55) is a class of exact solutions for equations (14). A family of solutions which depend on the arbitrary function $g_1(x)$ and where $g(x) = 8\Theta + 2k_1 x + k_2$ (k_1 and k_2 are arbitrary constants).

Table 1. Nontrivial exponents in (58) and compatibility.

	h	$g(x)$	$j(x)$	$k(x)$
(i)	-4	$k^2 g_x = -12 g\Theta_x$	$2k^2 j = g\Theta_x$	$kk_x = 4\Theta_x$
(ii)	2	k_1	Arbitrary	Arbitrary
(iii)	4	k_2	$j = -2\Theta_x/k_2$	k_3

When $\eta_\mu \neq 0$ the general compatibility equation is very complex, but it is possible as in the previous section, to consider some special case. The main goal of this section is to show that *all* the results contained in the paper by Goard and Broadbridge (1999) about the axisymmetric BL equations are obtainable by using the standard non-classical method and therefore the statement contained in this paper (see p 377) that the solutions displayed cannot be found by considering similarity reductions of (14) by the classical and non-classical method of group-invariant solutions is wrong.

We start considering the overdetermined system (see system (2.9) in Goard and Broadbridge (1999) with $\nu = 1$)

$$\begin{cases} \psi_\mu \psi_{\mu x} - \psi_x \psi_{\mu\mu} - 4\Theta_x = 2(\mu\psi_{\mu\mu})_\mu \\ 2(\mu\psi_{\mu\mu})_\mu = j(x)\psi_{\mu\mu\mu} + h(x)\psi_{\mu\mu} \end{cases} \tag{56}$$

which is clearly related to equation (14). By a direct integration of $(56)_2$, we obtain

$$\psi_\mu = g(x)(2\mu - j(x))^{h(x)/2} + k(x) \tag{57}$$

for this reason in the following we shall consider the system (51) where

$$\eta(x,\mu) = g(x)(2\mu - j(x))^{h/2} + k(x) \tag{58}$$

and h is constant (this is sufficient to recover the solutions in Goard and Broadbridge, (2001)).

By some lengthy, but straightforward computations, we obtain that the nontrivial compatibility of the system (51) is obtained only when $h = -4, 2, 4$. The details of the compatibility problem are reported synoptically in table 1.

In the case (i), the similarity solutions are given by

$$\psi(\mu, x) = -\frac{g(x)}{4\mu - 2j(x)} - k(x)\mu + f(x) \tag{59}$$

where

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{g(x)}{4k^3(x)}\left(k^2(x)\Theta_{xx} - 16\Theta_x^2\right) + 4. \tag{60}$$

In the case (ii), we obtain by reduction the following solution of (14):

$$\psi(\mu, x) = [(\mu - j(x))k_1 - k(x)]\mu + f(x). \tag{61}$$

First, let us take $k_1 = 0$, in this case in (61) the function $f(x)$ is arbitrary but

$$kk_x = 4\Theta_x \to k(x) = \pm\sqrt{8\Theta(x) + k_2}. \tag{62}$$

(Here k_2 is an arbitrary constant). Therefore, we recover the solutions in Goard and Broadbridge (2001, p 375), (see formula (2.21)).

On the other hand when $k_1 \neq 0$, it must be

$$f(x) = \frac{1}{4k_1}\left(k_1^2 j(x) + k^2(x) + 2k_1 j(x) - 8\Theta_x - 8k_1 x - 2k_4\right) \tag{63}$$

where $j(x)$ and $k(x)$ are arbitrary recovering as particular case the solutions (2.15) (Goard and Broadbridge 2001, p 374) and (2.19) of Goard and Broadbridge (2001, p 375).

In the case (iii), we recover the solution

$$\psi(\mu, x) = \left(\tfrac{4}{3}k_2\mu^2 - 2k_2 j(x)\mu - k_3 + k_2 j(x)\right)\mu + f(x) \tag{64}$$

where

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{1}{16k_3^2}\left(\Theta_x^2 - k_2k_3\right)\Theta_{xx} - 4. \tag{65}$$

The infinitesimal generator η in (58) is in some sense trivial because it does not depend on ψ; it is therefore natural to try the generalization

$$\eta = k(x)\psi + h(x, \mu) \tag{66}$$

where we require $k(x) \neq 0$. Because (66) is separable it is possible to use the same method of the previous section to solve the corresponding compatibility problem. If we consider the system (2.24) in Goard and Broadbridge (2001, p 376)

$$\begin{cases} \psi_\mu\psi_{\mu x} - \psi_x\psi_{\mu\mu} - 4\Theta_x = 2(\mu\psi_{\mu\mu})_\mu \\ 2(\mu\psi_{\mu\mu})_\mu = (k_1\mu + g(x))\psi_{\mu\mu\mu} + k(x)\psi_{\mu\mu} \end{cases} \tag{67}$$

we discover that (67) may be easily integrated obtaining

$$\psi(\mu, x) = g_1(x)\exp(g_2(x)\mu) + g_3(x)\mu + g_4(x). \tag{68}$$

From (68), by a simple differentiation, we find that ψ_μ is in correspondence with the infinitesimal generator

$$\eta(x, y, \psi) = g_2(x)\psi - g_2(x)g_3(x)\mu + g_3(x) - g_2(x)g_4(x) \tag{69}$$

a generator of the kind (66). Indeed it may be shown that this is a more general infinitesimal generator of the kind (66) for which the system (51) is compatible.

The direct introduction of (68) into (14) allows us to obtain

$$\Theta(x) = \tfrac{1}{2}\left(k_1^2x^2 - k_1k_2x + k_3\right) \qquad g_2(x) = k_1 \qquad g_3(x) = k_2 - k_1x \tag{70}$$

and

$$g_{4x} = -\frac{g_{1x}}{g_1}\left(2x - \frac{k_2}{k_1}\right) + 4 \tag{71}$$

a solution more general than the one reported in Goard and Broadbridge (2001).

5. Concluding remarks

The goal of this paper is to show that many solutions of the BL equations previously found by *new* direct reduction methods or *generalizations* of the concept of symmetry of a partial differential equation are indeed invariant solutions under the action of non-classical symmetries. This is an important point because it allows us a control of the completeness of the interesting result of Burde (1994).

Our computations are specific for the BL equations, but it is not a difficult task to extend our method to other kind of equations and to show that many of the generalizations which have been proposed in recent years of the usual reduction methods are indeed contained in the study of the overdetermined system $\mathcal{S}$.

For example, the result that the von Mises transformation may be found using the non-classical method is an important result. Indeed it allows us to recover and generalize all the ad hoc methods that have been introduced to derive similar transformations in more general frameworks as brine transport in porous media (van Dujin and Scotting 1998) and plasma oscillations (Numano 1975).

Obviously the results of this paper do not mean that other similarity reductions beyond the one obtainable via $\mathcal{S}$ are not possible. For example, because there is a possibility to rewrite (9) and (14) in a divergence form as

$$(\psi_{yy} + \psi_x \psi_y)_y - \left(\psi_y^2 - \Theta\right)_x = 0 \tag{72}$$

and

$$(2\mu\psi_{\mu\mu} + \psi_x \psi_\mu)_\mu - \left(\psi_\mu^2 - 4\Theta\right)_x = 0. \tag{73}$$

It is possible to find new similarity solutions related to nonlocal symmetries (see Saccomandi 1997), and similar considerations may be done if we consider generalized symmetries.

It is important to point out that the direct method of Burde (1994) is an interesting and effective way to find reductions of partial differential equations related to *true* weak symmetries. It is only a happenstance that in the case of the two-dimensional steady BL equations the system $\mathcal{S}$ reduces to the first-order differential system and therefore weak symmetries are not significant in determining similarity reductions. In more space dimensions and in the unsteady case, the method proposed by Burde may give effective invariant solutions under the action of true weak symmetries (see, for example, Burde (1995)). In this case the similarity reductions cannot be recovered by the Kruskal and Clarkson's direct method, because the invariance under a weak symmetry gives the possibility to reduce the (1) to a system of overdetermined ordinary differential equations and not to a single equation. For this reason the method of Burde remains a milestone in the development of effective direct methods to obtain exact solutions of nonlinear partial differential equations.

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